F. anal. 24.2.2025

- Slides, info (fredag)

A. Intermezzo: Interchanging integrals in $\mathbb{R}^2$

$$\int_{\mathbb{R}^d} f(x)\,dx = \lim_{R\to\infty} \int_{|x|\le R} f(x)\,dx \quad \text{(improper int.)}$$

Def. 1: $\underset{\text{moderate decrease}}{C_m(\mathbb{R}^d)} = \{ f \in C(\mathbb{R}^d) : \exists\, \varepsilon, A > 0 \text{ s.t. } |f(x)| \le \frac{A}{(1+|x|^2)^{\frac{d+\varepsilon}{2}}} \}$.

$f \in C_m(\mathbb{R}^d)$ (or $\in L^1(\mathbb{R}^d)$) $\Longrightarrow$ $\int_{\mathbb{R}^d} f(x)\,dx$ well-def.

$$\Big[\int_{\mathbb{R}^d} \underbrace{\frac{1}{(1+|x|^2)^{\frac{d+\varepsilon}{2}}}}_{\le 1,\ \le \frac{1}{|x|^{d+\varepsilon}}} dx \underset{\substack{\text{Coord.}\\ x = yr}}{\overset{\text{polar}}{=}} \lim_{R\to\infty} \int_{|y|=1} \int_0^R \frac{1}{(1+r^2)^{\frac{d+\varepsilon}{2}}} r^{d-1}\,dr\,dS(y)$$

$$\le \lim_{R\to\infty} \int_{|y|=1} dS(y) \Big(\int_0^1 1\, r^{d-1}\,dr + \int_1^R \underbrace{\frac{1}{r^{d+\varepsilon}} r^{d-1}}_{r^{-(1+\varepsilon)}}\,dr \Big)$$

$$= \lim_{R\to\infty} O(1)\big(O(1) + O(R^{-\varepsilon})\big) < \infty \Big]$$

Thm. 1: If $f \in C_m(\mathbb{R}^2)$, then

(i) $F(x_1) = \int_{\mathbb{R}} f(x_1, x_2)\,dx_2 \in C_m(\mathbb{R})$

(ii) $\iint_{\mathbb{R}\times\mathbb{R}} f(x_1,x_2)\,dx_1\,dx_2 = \int_{\mathbb{R}} \big(\int_{\mathbb{R}} f(x_1,x_2)\,dx_2 \big)\,dx_1$

Rem. 1: Fubini–Tonelli

(a) Thm. 1 holds replacing C_m by L^1

(b) (ii) holds for any $f \ge 0$ (measurable)

Pf.:

(i) $|F(x_1)| \le \left(\int_{|x_2|\le|x_1|} + \int_{|x_2|>|x_1|}\right) \dfrac{A}{(1+(x_1^2+x_2^2))^{\frac{2+\bar{\varepsilon}}{2}}}\, dx_2$

$$\le \frac{A}{(1+x_1^2)^{\frac{2+\bar{\varepsilon}}{2}}} \underbrace{\int_{|x_2|\le|x_1|} dx_2}_{=2|x_1|} + \underbrace{\int_{|x_2|>|x_1|} \frac{A}{(1+x_2^2)^{\frac{2+\bar{\varepsilon}}{2}}}\, dx_2}_{\overset{|x_1|\ge 1}{\le} 2\int_{x_1}^{\infty} \frac{A}{x_2^{2+\bar{\varepsilon}}}\, dx_2 \sim \frac{1}{x_1^{1+\bar{\varepsilon}}}}$$

$$\le \frac{C}{(1+x_1^2)^{\frac{1+\bar{\varepsilon}}{2}}}, \quad \tilde{\varepsilon} = \frac{\bar{\varepsilon}}{2}$$

(ii) Fix $\varepsilon > 0$. Let $I_N = [-N, N]$.

1) Take $N > 0$ s.t.

$$I_1 := \left| \left(\iint_{\mathbb{R}^2} - \iint_{I_N \times I_N} \right) f(x_1, x_2)\, dx_1 dx_2 \right| < \frac{\varepsilon}{2}$$

$$I_2 := \left| \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f\, dx_2 \right) dx_1 - \int_{I_N} \left(\int_{I_N} f\, dx_2 \right) dx_1 \right| < \frac{\varepsilon}{2}$$

Must chk, e.g.

$$I_2 \le \underbrace{\left| \int_{I_N^c} \Big(\underbrace{\int_{\mathbb{R}} f\, dx_2}_{F(x_1)} \Big) dx_1 \right|}_{\overset{(i)}{=} O(N^{-\bar{\varepsilon}})} + \left| \int_{I_N} \Big(\underbrace{\int_{I_N^c} f\, dx_2}_{\le \int_{|x_2|>N} \underbrace{\frac{A}{(1+x_1^2+x_2^2)^{\frac{2+\bar{\varepsilon}}{2}}}}_{\le \frac{A}{|x_2|^{2+\bar{\varepsilon}}}} dx_2} \Big) dx_1 \right|$$

$$\le O(N^{-\bar{\varepsilon}}) + \underbrace{\int_{I_N} dx_1}_{O(N)} \underbrace{\int_{|x_2|>N} \frac{A}{|x_2|^{2+\bar{\varepsilon}}}\, dx_2}_{O(N^{-1-\bar{\varepsilon}})}$$

$$\xrightarrow[N \to \infty]{} 0$$

[Chk: $I_1 \to 0$, $N \to \infty$]

2) Prev. results (Week 3, SS App.):

$$I_3 = \iint_{I_N \times I_N} f\, dx_1 dx_2 - \int_{I_N} \left(\int_{I_N} f\, dx_2 \right) dx_1 = 0$$

3) $\left| \iint_{\mathbb{R}^2} f\, dx_1 dx_2 - \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f\, dx_2 \right) dx_1 \right|$

$$\le I_1 + |I_3| + I_2 < \varepsilon \qquad \square$$

9:00

B. Convolutions and good kernels in $\mathbb{R}$

Convolution in $\mathbb{R}$:

(1) $(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy$

Rem. 2:

(a) $f, g \in C_m(\mathbb{R}),\ x \in \mathbb{R} \Rightarrow G(y) = f(x-y)g(y) \in C_m(\mathbb{R})$

[Exercise 7]

(b) By (a), (1) well-def. for $f, g \in C_m(\mathbb{R})$

(c) Usual properties of convolutions hold:

$f*g = g*f,\ (af+bg)*h = af*h + bg*h, \dots$

Lem. 1: $f, g \in C_m(\mathbb{R})$

$\Rightarrow \|f*g\|_\infty \le \|f\|_1 \|g\|_\infty,\ \|f*g\|_1 \le \|f\|_1 \cdot \|g\|_1$

Pf.:

$$\int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x-y)\,g(y)\,dy \right| dx \le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x-y)| \cdot |g(y)|\,dy \right) dx$$

$$\overset{\text{Thm. 1}}{=} \int_{\mathbb{R}} |g(y)| \underbrace{\left(\int_{\mathbb{R}} |f(x-y)|\,dx \right)}_{= \int_{\mathbb{R}} |f(x)|\,dx} dy = \|g\|_1 \cdot \|f\|_1$$

$\square$

Rem. 3: $f, g \in L^1(\mathbb{R}) \Rightarrow$ (1) well-def. (in L^1)

and Rem. 2 b-c, Lem. 1 still hold.

Good kernel $\{G_\delta\}_{\delta>0}$:

(i) $\int_{\mathbb{R}} G_\delta(x)\,dx = 1$

(ii) $\int_{\mathbb{R}} |G_\delta(x)|\,dx \le M$

(iii) $\forall \eta>0$, $\int_{|x|>\eta} |G_\delta(x)|\,dx \xrightarrow[\delta\to 0]{} 0$

Lem. 2:

$$f\in C_m(\mathbb{R}) \Rightarrow f*G_\delta \xrightarrow[\delta\to 0]{\text{unif.}} f$$

Leave in notes, skip in class

Pf.:

1) $f\in C_m(\mathbb{R}) \Rightarrow f$ unif. cont. [Exercise 7]

2) $|f*G_\delta(x) - f(x)| \overset{(i)}{=} \left|\int_{\mathbb{R}} (f(x-y)-f(x))\,G_\delta(y)\,dy\right|$

$$\overset{\eta\in(0,1)}{\le} \Big(\int_{|y|\le\eta} + \int_{|y|>\eta}\Big) |\cdots|$$

$$\le \underbrace{\sup_{|y|\le\eta}|f(x-y)-f(x)|}_{\le \sup_{|y|\le\eta}\sup_{x\in\mathbb{R}}|\cdots| \overset{1)}{\le} \omega_f(\eta) \xrightarrow[\eta\to 0]{} 0} \underbrace{\int_{|y|\le\eta}|G_\delta|}_{\overset{(ii)}{\le} M} + 2\|f\|_\infty \underbrace{\int_{|y|>\eta}|G_\delta|}_{\xrightarrow[\delta\to 0]{} 0\ (\eta \text{ fixed})}$$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ if η small and then δ small (enough) □

Pf.: $f\in C_m(\mathbb{R}) \Rightarrow f$ unif. cont. [Exercise 7]

Rest of pf. as for $f\in C([0,1])$ (prev., see notes and SS) □

Cor. 1: $K_\delta(x) = \frac{1}{\sqrt{\delta}} e^{-\pi\frac{x^2}{\delta}}$ (Gaussians), $f\in C_m(\mathbb{R})$

$$\Rightarrow K_\delta * f \xrightarrow[\delta\to 0]{\text{unif.}} f$$

C. Fourier inversion in $S(\mathbb{R})$

Prop. 1 The multiplication formula

$$f, g \in S(\mathbb{R}) \Rightarrow \int_{-\infty}^{\infty} f(x)\,\hat{g}(x)\,dx = \int_{-\infty}^{\infty} \hat{f}(y)\,g(y)\,dy$$

Pf.:

$$f, g \in S(\mathbb{R}) \Rightarrow \hat{f}, \hat{g} \in S(\mathbb{R}), \quad S(\mathbb{R}) \subset C_m(\mathbb{R})$$

$$\int_{-\infty}^{\infty} f(x) \left(\int_{-\infty}^{\infty} g(\xi)\, e^{-2\pi i x \xi}\, d\xi\right) dx$$

$$\overset{\text{Thm. 1}}{=} \iint_{\mathbb{R}^2} f(x)\, g(\xi)\, e^{-2\pi i x \xi}\, d\xi\, dx$$

$$\overset{\text{Thm. 1}}{=} \int_{\mathbb{R}} g(\xi) \left(\int_{\mathbb{R}} f(x)\, e^{-2\pi i x \xi}\, dx\right) d\xi \qquad \square$$

Thm. 2: (F.-inversion)

$$f \in S(\mathbb{R}) \Rightarrow f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi \quad \left(\text{"} = \mathcal{F}^{-1}[\hat{f}](x) \text{"}\right)$$

Pf.: 1) Chk: $K_1 = \hat{K}_1 = \hat{\hat{K}}_1 \Rightarrow \hat{\hat{K}}_\delta = \widehat{(e^{-\pi \delta \xi^2})} = K_\delta$ [Exercise 8]

2) $f(0) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, d\xi$:

$$\underbrace{\int_{-\infty}^{\infty} f(x) \underset{\hat{\hat{K}}_\delta}{\underbrace{K_\delta(x)}_{\overset{1)}{=}}}\, dx}_{f * K_\delta(0)} \overset{\text{Prop. 1}}{=} \int_{-\infty}^{\infty} \hat{f}(\xi)\, \hat{K}_\delta(\xi)\, d\xi$$

$\hat{K}_\delta(\xi) \overset{\text{last time}}{=} e^{-\pi \delta \xi^2} \xrightarrow[\delta \to 0]{\text{unif.}} 1$ (chk!)

$f * K_\delta(0) \xrightarrow[\delta \to 0]{\text{Cor. 1}} f(0)$

$\int_{-\infty}^{\infty} \hat{f}(\xi)\, \hat{K}_\delta(\xi)\, d\xi \xrightarrow{\delta \to 0} \int_{-\infty}^{\infty} \hat{f}(\xi)\, d\xi$ [Exercise 8]

3) General result: $g(y) := f(x+y)$

$$f(x) = g(0) \overset{2)}{=} \int_{-\infty}^{\infty} \hat{g}(\xi)\, d\xi \underset{\mathcal{F}_y[f(x+y)] = e^{2\pi i x \xi} \hat{f}(\xi)}{=} \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi \qquad \square$$

F. anal. 27.2.2025

- Slides, info: no ex. class tomorrow

A. The Fourier inversion

Def.: $\mathcal{F}^*[f](x) := \int_{-\infty}^{\infty} f(\xi) e^{2\pi i x \xi} d\xi$ $\quad (\mathcal{F}[f](\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx)$

Lem. 1:

(a) $g \in C_m(\mathbb{R}) \Rightarrow \mathcal{F}^*[g]$ well-def.

(b) $g \in C_m(\mathbb{R})$

$\Rightarrow \mathcal{F}^*[g(y)](x) = \mathcal{F}[g(y)](-x) = \mathcal{F}[g(-y)](x)$

(c) $g \in S(\mathbb{R}) \Rightarrow \mathcal{F}^*[g] \in S(\mathbb{R})$

Pf.:

(a) as for $\mathcal{F}$

(b) $\int_{\mathbb{R}} g(y) e^{+2\pi i x y} dy = \int_{\mathbb{R}} g(y) e^{-2\pi i (-x) y} dy$

$\overset{\bar{y}=-y}{=} \int_{\mathbb{R}} g(-\bar{y}) e^{-2\pi i \bar{y} x} d\bar{y}$

(c) $\cancel{g \in S(\mathbb{R}) \Rightarrow}$ $\mathcal{F}^*[g](x) \overset{b)}{=} \mathcal{F}[g](-x) \in S(\mathbb{R})$ $\square$
since $g \in S(\mathbb{R})$

Rem. 1: (a), (b) ok for $g \in L^1(\mathbb{R})$

Thm. 1: $f, g \in S(\mathbb{R})$

(a) $f(x) = \mathcal{F}^*[\mathcal{F}[f]](x)$

(b) $g(y) = \mathcal{F}[\mathcal{F}^*[g]](y)$

Pf.:

(a) proved last time

(b) $\mathcal{F}[\mathcal{F}^*[g](x)](y) \overset{\text{Lem. 1b}}{=} \mathcal{F}[\mathcal{F}[g](-x)](y)$

$\overset{\text{Lem 1b}}{=} \mathcal{F}^*[\mathcal{F}[g](x)](y) \overset{(a)}{=} g(y)$ $\square$

Hence $\mathcal{F}^* \circ \mathcal{F} = I$, $\mathcal{F} \circ \mathcal{F}^* = I$ on $S(\mathbb{R})$,

and $\mathcal{F}^* = \mathcal{F}^{-1}$ on $S(\mathbb{R})$.

Cor. 1: $\mathcal{F}: S(\mathbb{R}) \to S(\mathbb{R})$ is invertible

with inverse $\mathcal{F}^{-1} = \mathcal{F}^*$.

B. The Plancherel identity

Similar to Parseval's identity for F-series.

$L^2(\mathbb{R}) = \{ f : f \text{ measurable}, \ \|f\|_2 < \infty \}$

Inner-prod. sp. with

$(f, g) = (f, g)_2 := \int_{\mathbb{R}} f(x) \overline{g(x)} \, dx,$

$$\|f\| = \|f\|_2 = (f,f)^{\frac{1}{2}} = \left(\int |f|^2 dx\right)^{\frac{1}{2}}.$$

Lem. 2: $f \in S(\mathbb{R}) \Rightarrow \|f\|_2 < \infty$ and $f \in L^2(\mathbb{R})$

Pf.: $\int |f|^2 dx \leq \underbrace{\sup_{x\in\mathbb{R}} (1+|x|^2)|f(x)|^2}_{\substack{<\infty \\ f\in S(\mathbb{R})}} \underbrace{\int \frac{1}{1+|x|^2} dx}_{<\infty}$ $\square$

Thm. 1: (Plancherel) If $f,g \in S(\mathbb{R})$, then

(i) $\|f\|_2 = \|\hat{f}\|_2$

(ii) $(f,g)_2 = (\hat{f},\hat{g})_2$ [see p. 81 for pf.]

Rem. 1:

(i) $\Rightarrow$ $\mathcal{F}$ isometry on $S(\mathbb{R})$ w.r.t. $\|\cdot\|_2$,

i.e. do not change $\|\cdot\|_2$-norm of $f \in S(\mathbb{R})$.

Pf. needs properties of $*$ in $S(\mathbb{R})$:

Prop. 1:
$f*g = g*f$
$\partial_x^k(f*g) = f*(\partial^k g)$
Prop. 2
$f*g \in S$
$\widehat{f*g} = \hat{f}\cdot\hat{g}$

Prop. 1: $f,g \in S(\mathbb{R})$

(a) $f*g \in S(\mathbb{R})$

(b) $f*g = g*f$

(c) $\widehat{f*g}(\xi) = \hat{f}(\xi)\cdot\hat{g}(\xi)$

Pf.:

(a) $\sup_{x\in\mathbb{R}} \underbrace{|x|^l}_{\leq (|y|+|x-y|)^l} |f(x-y)| \leq \sup_{x:\,|y|\leq|x-y|} (2|x-y|)^l |f(x-y)| + \sup_{x:\,|x-y|\leq|y|} (2|y|)^l |f(x-y)|$

$\leq \sup_{z\in\mathbb{R}} 2^l |z|^l |f(z)| + 2^l |y|^l \|f\|_\infty \overset{f\in S(\mathbb{R})}{\leq} C_l(1+|y|^l)$

Hence

$$\sup_x |x^l (f*g)(x)| \le \sup_x \int \underbrace{\underbrace{|x|^l |f(x-y)|}_{\le c_l(1+|y|^l)} \cdot |g(y)|}_{\le \tilde{c}_l \frac{1}{1+y^2}} \, dy < \infty \quad g \in S(\mathbb{R})$$

$$\tilde{c}_l = c_l \sup_{y\in\mathbb{R}} (1+|y|^l)(1+y^2)|g(y)|$$

11:00

Chk.: $\partial^k (f*g) = f * (\partial^k_x g)$ [Exercise 8]

$$\sup_x |x^l \cdot \partial^k (f*g)(x)| \le c_l \int_{\mathbb{R}} (1+|y|^l) |\partial^k_x g(y)| \, dy < \infty \quad g \in S(\mathbb{R})$$

and $(f*g) \in S(\mathbb{R})$.

(b) $$\int_{\mathbb{R}} f(x-y) g(y)\, dy \underset{du=-dy}{\overset{u=x-y}{=}} \int_{-\mathbb{R}} f(u) g(x-u)(-du) = \int_{\mathbb{R}} f(u) g(x-u)\, du$$

(c) Chk.: $f, g \in S(\mathbb{R}) \Rightarrow F(x,y) = f(x-y) g(y) e^{-2\pi i x \xi} \in C_m(\mathbb{R}^2)$

[By pf. of a), $|x|^3 |f(x-y)| \cdot |g(y)| \le c\,(1+|y|^3) \cdot |g(y)| \le C_l < \infty$

$|y|^3 |f(x-y)| \cdot |g(y)| \le \|f\|_\infty |y|^3 |g(y)| \le \tilde{C}_l < \infty$]

$\Rightarrow |F(x,y)| \le \frac{C}{1+|x|^3+|y|^3}$ and $F \in C_m(\mathbb{R}^2)$]

By interchanging order of integr.

$$\mathcal{F}[f*g](\xi) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x-y) g(y)\, dy \right) e^{-2\pi i x \xi} dx$$

$$= \iint_{\mathbb{R}^2} F(x,y)\, dx dy$$

$$= \int_{\mathbb{R}} g(y) e^{-2\pi i y \xi} \underbrace{\left(\int_{\mathbb{R}} f(x-y) e^{-2\pi i (x-y)\xi} dx \right)}_{\hat{f}(\xi)} dy$$

$$= \hat{g}(\xi) \cdot \hat{f}(\xi) \qquad \square$$

Pf. of Thm. 1 (Plancherel):

(i)

1) $f^b(x) := \overline{f(-x)}$.

$$\widehat{f^b}(\xi) \overset{\text{chk}}{=} \int_{\mathbb{R}} \overline{f(-x)}\, \overline{e^{-2\pi i(-x)\xi}}\, d\xi \underset{\int \bar{f} = \overline{\int f}}{=} \overline{\hat{f}(\xi)}$$

$$\overline{e^{-2\pi i(-x)\xi}} = e^{-2\pi i x \xi}$$

2) $h := f * f^b$

$$\hat{h} \overset{\text{Prop. 1c}}{=} \hat{f} \cdot \widehat{f^b} \overset{1)}{=} \hat{f}\,\overline{\hat{f}} = |\hat{f}|^2$$

$$h(0) = \int_{\mathbb{R}} f(0-y) f^b(y)\, dy = \int_{\mathbb{R}} f(-y)\overline{f(-y)}\, dy \underset{\text{chk}}{\overset{x=-y}{=}} \int_{\mathbb{R}} |f(x)|^2 dx$$

3) Fourier inversion (Thm. 1):

$$\|f\|_2^2 \overset{2)}{=} h(0) \overset{\text{Thm. 1a}}{=} \mathcal{F}^{-1}[\hat{h}](0) \overset{\text{def. } \mathcal{F}^{-1}}{=} \int_{-\infty}^{\infty} \hat{h}(\xi)\, e^0 d\xi \overset{2)}{=} \int_{-\infty}^{\infty} |\hat{f}|^2 d\xi$$

□

(ii) From a), since

$$(f, g)_2 \overset{\text{chk.}}{=} \frac{1}{4}\left(\|f+g\|_2^2 + \|f-g\|_2^2 + i\left(\|f+ig\|^2 - \|f-ig\|^2\right)\right)$$

$$\overset{a)}{=} \frac{1}{4}\left(\|\hat{f}+\hat{g}\|_2^2 + \cdots\right) = (\hat{f}, \hat{g})_2$$

□